Journal of Approximation Theory 101, 156–159 (1999)
Article ID jath.1999.3356, available online at http://www.idealibrary.com on IDEAL®

NOTE

Preservation of p-Continuity by Bernstein-Type Operators[1]

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Communicated by Dany Leviatan

Received March 10, 1998; accepted January 15, 1999

We show that Bernstein-type operators preserving monotonicity (and a fairly general additional assumption) also preserve the p-continuity. © 1999 Academic Press

Generally speaking, a Bernstein-type operator L on an interval I of the real line is a positive linear operator acting on real functions defined on I and having the form

$$Lf(x) = \int_I f\, dv_x, \qquad x \in I, \quad f \in \mathscr{L}, \tag{1}$$

where v_x is a Borel probability measure on I, and $\mathscr{L}$ stands for the domain of L, i.e., the set of all real measurable functions on I for which the right-hand side in (1) makes sense. Obviously, $\mathscr{L}$ includes all the real measurable bounded functions on I. If all the probability measures v_x are discrete, the measurability condition can even be dropped.

The celebrated Bernstein operator producing the polynomials of the same name is the most paradigmatic example of this kind of operator. Other examples well known in the literature are the Schoenberg variation diminishing operator, the Szász operator, the Baskakov operator, the gamma operator, etc. By the Riesz representation theorem, if I is compact, every positive linear operator $L: C(I) \to C(I)$ such that $L(1) \equiv 1$ has the form (1).

The preservation properties of the operators of the above type have been actively investigated during the past decade (see, for instance, [1–3] and

[1] Research supported by DGICYT Grant PB95-0809 and by Grant BFI96.014 of the Basque Government.

0021-9045/99 $30.00

the references therein). In this note, we obtain a fairly general result concerning preservation of p-continuity. Before stating it in a precise way, we introduce some definitions and notations. For $p \in [1, \infty]$, and any real function f defined on I, the modulus of p-continuity $\omega_p(f; \cdot)$ is defined by

$$\omega_p(f; \delta) := \begin{cases} \sup(\sum_{i=1}^n |f(x_i) - f(y_i)|^p)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup \max_{1 \leqslant i \leqslant n} |f(x_i) - f(y_i)| & \text{if } p = \infty. \end{cases} \qquad \delta \geqslant 0,$$

where the supremum is taken over all the finite sequences of pairwise disjoint subintervals of I, (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), such that $\sum_{i=1}^n (y_i - x_i) \leqslant \delta$. We say that f is p-continuous if $\omega_p(f; \delta) \to 0$, as $\delta \to 0$. Observe that ω_∞ is the usual modulus of continuity, so that ∞-continuity is nothing but uniform continuity. On the other hand, 1-continuity is nothing but absolute continuity. If $1 \leqslant p < q \leqslant \infty$, we have $\omega_q(f; \cdot) \leqslant \omega_p(f; \cdot)$, and, therefore, p-continuity implies q-continuity. The set of all real functions on I which are p-continuous (resp. nondecreasing) will be denoted by $\mathscr{A}_p$ (resp. $\mathscr{I}$). Finally, e_1 will stand for the monomial $e_1(x) := x$. It is readily seen that the assumption $e_1 \in \mathscr{L}$ implies that $\mathscr{A}_\infty \subset \mathscr{L}$ and, therefore, $\mathscr{A}_p \subset \mathscr{L}$, for every $p \in [1, \infty]$.

We ennunciate the following.

THEOREM. *Assume that the following two conditions are fulfilled*:

(a) *L preserves monotonicity, i.e., $L(\mathscr{I} \cap \mathscr{L}) \subset \mathscr{I}$.*

(b) *$e_1 \in \mathscr{L}$ and $Le_1 \in \mathscr{A}_1$.*

Then, for all $p \in [1, \infty)$, $f \in \mathscr{L}$ and $\delta \geqslant 0$, we have

$$\omega_p(Lf; \delta) \leqslant 2\, \omega_p(f; \omega_1(Le_1; \delta)). \tag{2}$$

Therefore, L preserves the p-continuity; i.e., $L(\mathscr{A}_p) \subset \mathscr{A}_p$.

Proof. Assume that the set $\mathscr{P}(I)$ of all Borel probability measures on I is endowed with the usual stochastic order (cf. [4]). Then, the assumption (a) can be paraphrased by saying that the map (from I into $\mathscr{P}(I)$) $x \mapsto \nu_x$ is nondecreasing. Assumption (a) is also equivalent (extend in an obvious way the proof of Theorem 1.A.1 in [4]) to the fact that there exists an I-valued stochastic process $\{Z^x : x \in I\}$ such that, for each $x \in I$, the probability distribution of the random variable Z^x is ν_x, and

$$Z^x \leqslant Z^y \quad \text{a.s.} \qquad \text{for all} \quad x, y \in I \quad \text{with} \quad x \leqslant y. \tag{3}$$

Therefore, (1) can be written in the form

$$Lf(x) = Ef(Z^x), \tag{4}$$

where E denotes mathematical expectation; in particular, $Le_1(x)=EZ^x$, and the assumption $e_1 \in \mathscr{L}$ means that the above stochastic process is integrable.

Let $\delta>0$, and let $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ be a finite sequence of pairwise disjoint subintervals of I such that $\sum_{i=1}^n (y_i - x_i) \leqslant \delta$. By (3), we have that the random subintervals $(Z^{x_1}, Z^{y_1}), (Z^{x_2}, Z^{y_2}), ..., (Z^{x_n}, Z^{y_n})$ are a.s. pairwise disjoint. Using (4), we therefore have, for all $p \in [1, \infty)$ and $f \in \mathscr{L}$,

$$\begin{aligned}\left(\sum_{i=1}^n |Lf(x_i)-Lf(y_i)|^p\right)^{1/p} &= \left(\sum_{i=1}^n |Ef(Z^{x_i})-Ef(Z^{y_i})|^p\right)^{1/p}\\ &\leqslant E\left(\sum_{i=1}^n |f(Z^{x_i})-f(Z^{y_i})|^p\right)^{1/p}\\ &\leqslant E\omega_p(f; \Delta), \qquad (5)\end{aligned}$$

where $\Delta := \sum_{i=1}^n (Z^{y_i} - Z^{x_i})$. Since $\omega_p(f; \cdot)$ is subadditive, we can write, for any $h>0$,

$$E\omega_p(f; \Delta) \leqslant \left(1+\frac{E\Delta}{h}\right) \omega_p(f; h). \qquad (6)$$

From assumption (b), we get

$$E\Delta = \sum_{i=1}^n (EZ^{y_i} - EZ^{x_i}) \leqslant \omega_1(Le_1; \delta).$$

On taking $h=\omega_1(Le_1; \delta)$ in (6), we therefore obtain

$$E\omega_p(f; \Delta) \leqslant 2\, \omega_p(f; \omega_1(Le_1; \delta)). \qquad (7)$$

From (5) and (7), the inequality (2) immediately follows, and the proof of the theorem is complete.

Remark. In all the usual examples, Le_1 is a linear function, and assumption (b) is automatically fulfilled. On the other hand, it is well known that the theorem above also holds in the case $p=\infty$ (see [1, 3]). We also refer to [3] for a different approach to the problem of preservation of absolute continuity.

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